EXPLICIT SOLUTIONS FOR THE EXIT PROBLEM FOR A CLASS OF LÉVY PROCESSES. APPLICATIONS TO THE PRICING OF DOUBLE BARRIER OPTIONS

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ABSTRACT. Lewis and Mordecki have computed the Wiener-Hopf factorization of a Lévy process whose restriction on $]0, +\infty[$ of their Lévy measure has a rational Laplace transform. That allows to compute the distribution of $(X_t, \inf_{0 \le s \le t} X_s)$. For the same class of Lévy processes, we compute the distribution of $(X_t, \inf_{0 \le s \le t} X_s, \sup_{0 \le s \le t} X_s)$ and also the behavior of this triple at certain stopping time, like the first exit time of an interval containing the origin. Some applications to the pricing of double barrier options with or without rebate are evocated.

1. INTRODUCTION

There are very few examples of Lévy process for which the so-called "exit problem" can be explicitly solved (see [KK05] , [R90], [R72]). We present here this explicit solution for a class of Lévy processes which has been introduced by Lewis A.L. and Mordecki E.[LM08]; that is the class of the Lévy processes whose restriction on $]0, +\infty[$ of their Lévy measure has a rational Laplace transform. This happens when this restriction is a finite linear combination of exponential or gamma distributions. Lewis A.L and Mordecki E. [LM08] (see also Asmussen, S. Avram, F. and Pistorius M.R. [AAP04]) have computed the so-called Wiener-Hopf factorization of these Lévy process. That brings to closed forms for the distribution of the maximum of the process before an independent exponential time and for the joint distribution of (T^x, X_{Tx}) where T^x is the first time where the Lévy process X crosses upward a level x and X_{T^x} its position at that time.

As an application, when adopting the exponential Lévy model, $Y_t = Y_0.e^{X_t}$, for a financial asset Y_t with X_t of the preceding form (see for example [CT03]), an immediate consequence of the preceding results is the computation of the (temporal Laplace transform) price of the double barrier option with this underlying asset. That is, an european option which is activated (in) or desactivated (out) when the asset $(Y_t)_{t \in [0,T]}$ cross up (down) a barrier H before the time of maturity T. This application follows the computation of the price of the simple barrier option, by the Wiener-Hopf factorization (see [AAP04]).

In this paper, for the same class of Lévy processes, we solve the exit problem. More precisely, we give closed form of the joint distribution of the minimum and maximum of the process before an independent exponential time and among other

behaviors of the process at certain stopping times, we give the Laplace transform of the joint distribution of $(T_a^b, X_{T_a^b})$ where T_a^b is the first time that the process leaves a bounded interval $[-a, b]$ containing the origin and $X_{T_a^b}$ is its position at that time.

As an application, we mention how one can deduce the price of the double barrier option, in or out, that means that the option is activated or desactived if the asset crosses up a barrier b or down a barrier a in the exponential Lévy model.

The paper is organized as follow : In section 2, we recall the general results on Wiener-Hopf factorization, we give a shortened proof of the results of Lewis-Mordecki and settle few other preliminary results. For that, we use exclusively elementary complex analysis arguments. After introducing few more notations in section 3, we give in section 4, 5, 6, 7 and 8, all the results on the fluctuations of our class of Lévy processes. In section 9, we recall the main result of [F10] which is the main tool of this work. The proofs follow in section 10, 11, 12, 13.

These results, in the symmetric case, are related to what is called "Bargmann equations" in the litterature on inverse problems of the spectral theory (see for example [F63]).

2. The assumption and the Wiener-Hopf factorization

We suppose that X is a real Lévy process possibly killed at an independent expo-nantial time and we denote ζ the life time of X.

Let ϕ be the Lévy exponent of X, so that the identity

$$\mathbf{E}(e^{-iuX_t}1_{t<\zeta}) = e^{-t\phi(iu)}$$

is fulfilled for every time t and every imaginary number $iu \in i\mathbf{R}$, ϕ is continuous on $i\mathbf{R}$ and $\phi(0)$ is the rate of the exponential distribution of the life time ζ,

$$\phi(0) = 0 \quad \text{if and only if } \zeta = +\infty \text{ a.s. ("}X \text{ does not die").}$$

We now work under Lewis-Mordecki's assumption. It is based on the following rather obvious fact.

Proposition 2.1. *The conditions below are equivalent*
(i) The Lévy measure of X, π, is of the following form

$$\pi(dy) = \sum_{j=1}^{n} c_j \frac{y^{n_j}}{n_j!} e^{-\gamma_j y} dy \qquad on \quad]0, +\infty[$$

$$n \in \mathbf{N}, n_j \in \mathbf{N}, \Re(\gamma_j) > 0, c_j \in \mathbf{C} \qquad (j = 1, \dots, n)$$

(ii) The exponent ϕ is of the form

$$\phi(iu) = \phi^-(iu) + \frac{P}{Q}(iu),$$

where ϕ^- is the exponent of a Lévy process without positive jumps, P and Q are polynomials and Q has all its roots on the complex half plane $\{\lambda; \Re(\lambda) < 0\}$.

Proof. Assume (i). Notice that the compound Poisson process with Lévy measure

$$\pi(dy) = \sum_{j=1}^{n} c_j \frac{y^{n_j}}{n_j!} e^{-\gamma_j y} 1_{y>0} dy$$

has Lévy exponent

$$\sum_{j \in J} c_j \big[\frac{1}{(\gamma_j)^{n_j+1}} - \frac{1}{(\lambda + \gamma_j)^{n_j+1}} \big] =: \frac{P(\lambda)}{Q(\lambda)},$$

The function $\phi - \frac{P}{Q}$ is then the Lévy exponent of a Lévy process without positive jumps. This establishes (ii).

Conversely, assume (ii). Since ϕ and ϕ^- are exponents of Lévy processes, we know that $\phi(iu) = O(u^2)$ and $\phi^-(iu) = O(u^2)$ (see proposition 2 chapter 1 of [B96]), thus $\frac{P}{Q}(iu)$, construed as a rational function of u, is $O(u^2)$. Consequently $\deg P \leq \deg Q + 2$.

Now, write $\frac{P}{Q}(iu)$ in its fractional expansion

$$\frac{P}{Q}(iu) = au^2 + bu + c + \sum_{j=1}^{n} \frac{k_j}{(iu + \gamma_j)^{n_j+1}}$$

Then $\frac{P}{Q}(iu)$ is the Fourier transform of the Schwartz distribution (here, δ stands for the Dirac mass at 0);

$$a\delta'' - ib\delta' + c\delta + \sum_{j=1}^{n} k_j \frac{x^{n_j}}{n_j!} e^{-\gamma_j x} 1_{x>0} dx$$

On the other hand, one can deduce from the Lévy Kinchin formula applied to ϕ that $\phi(iu)$ is the Fourier transform of a distribution whose restriction to $]0, +\infty[$ is $-1_{x>0}\pi(dx)$ (see chapter 5 of [V02] for example) where π is the Lévy measure. From this, we see that

$$1_{x>0}\pi(dx) = -\sum_{j=1}^{n} k_j \frac{x^{n_j}}{n_j!} e^{-\gamma_j x} 1_{x>0} dx$$

This finishes the proof.

$\square$

The following is a famous result (see proposition 2, chapter I of [B96]) which applies to any Lévy process and which will be of use later .

Lemma 2.2. *There exists an exponent of a subordinator ψ and an exponent of the opposite of a subordinator $\check{\psi}$, such that*

$$\psi.\check{\psi} = \phi \qquad on \qquad i\mathbf{R}$$

Such a couple $(\psi, \check{\psi})$ is unique up to a multiplicative constant.

The functions ψ and $\check{\psi}$ will be refered to as the "positive" (for ψ) and "negative" (for $\check{\psi}$) Wiener -Hopf factors of the exponent ϕ, unlike the ordinary uses which affect these expressions to the functions $\frac{\psi(0)}{\psi}$ and $\frac{\check{\psi}(0)}{\check{\psi}}$.

ASSUMPTION : In the rest of this paper, conditions (i) and (ii) of proposition 2.1 will be assumed to be satisfied and this assumption will be call Assumption 2.1 .

Property (ii) of proposition 2.1 and the fact that ϕ^- has an holomorphic extension on the half plane $\{\Re(\lambda) < 0\}$ (see chapter 7 of [B96]) imply that ϕ has a meromorphic extension on this half plane, and we we will still denote by ϕ this extension.

We will denote
$$-\gamma_1, -\gamma_2, ..., -\gamma_n$$
the poles of ϕ repeated according to their multiplicity. In particular, $\deg Q = n$ when the fraction $\frac{P}{Q}$ is irreductible, in the expression of condition (ii) of proposition 2.1 ($\phi = \phi^- + \frac{P}{Q}$) and we will denote

$$n_j := \sharp\{k < j; \gamma_k = \gamma_j\} \qquad j = 1, \ldots, n$$

Notice that the n_j are all zero if and only if all the poles of ϕ on the half plane $\{Re(\lambda) < 0\}$ are simple, or equivalently, if and only if the Lévy measure is a (not necessarily positive) combination of exponential distributions.

Theorem 2.3. ψ *is of the form :*

$$\psi(\lambda) = \psi_\infty \frac{\Pi_1^m(\lambda + \beta_i)}{\Pi_1^n(\lambda + \gamma_j)},$$

where ψ_∞ is a positive constant.
If $\lim_{\lambda \to -\infty} \frac{\phi(\lambda)}{\lambda} \in]0, +\infty]$ then $m = n + 1$. Otherwise, $\lim_{\lambda \to -\infty} \frac{\phi(\lambda)}{\lambda} \in] -\infty, 0]$ and $m = n$.
More over, $\{-\beta_1, \ldots, -\beta_n\}$ is the set of the roots of ϕ lying on the half plane $\{\Re(\lambda) < 0\}$ (repeated according to their multiplicity) together with 0 if $\phi(0) = 0$ and $\phi'(0) \geq 0$.

Proof. Notice first that, thanks to Lévy Kintchin formula, $\phi(\lambda)$ does not vanish for λ on the imaginary axis except possibly for $\lambda = 0$, ϕ is a meromorphic funtion on $\{\Re(\lambda) < 0\}$ and ϕ is continuous by the left at each point of the imaginary axis $i\mathbf{R}$. On the other hand, the function ψ is an exponent of a subordinator, thus it is analytic on the open half plane $\{\Re(\lambda) > 0\}$ and continuous on the closed half plane $\{\Re(\lambda) \geq 0\}$ (see [B96] chapter 3 for example). Also, the identity $\psi\check{\psi} = \phi$ on $i\mathbf{R}$ implies that $\psi = \frac{\phi}{\check{\psi}}$ on $i\mathbf{R}$. Thus, since ϕ is meromorphic and $\check{\psi}$ holomorphic on $\{\Re(\lambda) < 0\}$, we see that ψ has a meromorphic extension on $\{\Re(\lambda) < 0\}$. We denote again ψ this extension.
More over $\psi = \frac{\phi}{\check{\psi}}$ is continuous by the left on the axis $i\mathbf{R}$, except possibly at $\lambda = 0$ when $\check{\psi}(\lambda) = 0$.
Then ψ is meromorphic on $\mathbf{C}$ except possibly at 0. However, because $\check{\psi}$ is the exponent of the opposite of a subordinator, we have (see page 73 of [B96])

$$\lim_{\lambda \to 0; \Re(\lambda) < 0} \frac{\lambda}{\check{\psi}(\lambda)} \to] -\infty, 0],$$

then

$$\lim_{\lambda \to 0; \Re(\lambda) < 0} \lambda. \frac{\phi(\lambda)}{\check{\psi}(\lambda)} = \lim_{\lambda \to 0; \Re(\lambda) < 0} \lambda\psi(\lambda) = 0.$$

Also, ψ is continuous by the right at the point 0, then $\lim_{\lambda \to 0; \Re(\lambda) > 0} \lambda\psi(\lambda) = 0$ and we deduce that ψ is again holomorphic at 0.
In conclusion, ψ is meromorphic on $\mathbf{C}$ and its poles, necessarily in the half plane $\{\lambda; \Re(\lambda) < 0\}$, are the same as those of ϕ (because of the identity $\psi\check{\psi} = \phi$ and because $\check{\psi}$ is holomorphic on $\{\lambda; \Re(\lambda) < 0\}$). Therefore, these poles are the $-\gamma_j$.
Now, write

$$\psi(\lambda) = \frac{\Psi(\lambda)}{\Pi_1^n(\lambda + \gamma_j)},$$

where Ψ is an entire function. Notice the limit, valid for any exponent of a subordinator (see proposition 2 of Chapter 1 of [B96])

$$\lim_{\lambda \to +\infty} \frac{\psi(\lambda)}{\lambda} \in [0, +\infty[$$

Then

(2.1) $$\Psi(\lambda) = O(\lambda^{n+1}) \qquad (\Re(\lambda) \to +\infty)$$

And, since

$$\psi(\lambda) = \frac{\Psi(\lambda)}{\Pi_1^n(\lambda + \gamma_j)} = \frac{\phi^-(\lambda) + \frac{P(\lambda)}{Q(\lambda)}}{\check{\psi}(\lambda)},$$

$\phi^-(\lambda)$ and $\frac{P(\lambda)}{Q(\lambda)}$ are $O(\lambda^2)$ on $\{\Re(\lambda) < 0\}$ (see proof of proposition 2.1) and $\frac{1}{\check{\psi}(\lambda)}$ is bounded (by $\frac{1}{\check{\psi}(-1)}$) on $\{\Re(\lambda) < -1\}$, because it is the Laplace transform of a measure supported by $]-\infty, 0]$ (see chapter 3 of [B96] for example) and $\deg P \le \deg Q + 2$, we deduce that

(2.2) $$\Psi(\lambda) = O(\lambda^{n+2}) \qquad (\Re(\lambda) \to -\infty)$$

When joining property (2.1) and (2.2), Ψ is a polynomial with degree at most $n+1$. Moreover, since $\frac{1}{\psi(\lambda)}$ is bounded for $\lambda \to +\infty$, then Ψ has a degree at least n.
Thus Ψ is a polynomial and its degree is n or $n + 1$.
If Ψ is a polynomial of degree $n + 1$ then

$$\lim_{\lambda \to -\infty} \frac{\psi(\lambda)}{\lambda} \in]0, +\infty[\quad \text{and} \quad \lim_{\lambda \to -\infty} \check{\psi}(\lambda) \in]0, +\infty]$$

(this last limit is valid for any exponent of the opposite of a subordinator). Since $\phi = \psi\check{\psi}$, we obtain

$$\lim_{\lambda \to -\infty} \frac{\phi(\lambda)}{\lambda} \in]0, +\infty].$$

And, if Ψ is a polynomial of degree n, then

$$\lim_{\lambda \to -\infty} \psi(\lambda) \in]0, +\infty[\quad \text{and} \quad \lim_{\lambda \to -\infty} \frac{\check{\psi}(\lambda)}{\lambda} \in]-\infty, 0],$$

(this last limit is valid for any exponent of the opposite of a subordinator) and we obtain

$$\lim_{\lambda \to -\infty} \frac{\phi(\lambda)}{\lambda} \in]-\infty, 0]$$

Finaly, $\check{\psi}$ does not vanish on the left half plane $\{\Re(\lambda) < 0\}$ because it is the exponent of the opposite of a subordinator and so, the roots of ψ, which are necessary on the left half plane $\{Re(\lambda) < 0\}$, except possibly 0 (because the exponent of a subordinator doesn't vanish in the half plane $\{Re(\lambda) > 0\}$), are the one of ϕ. Let us study when 0 is a root of ψ.
The function $\check{\psi}$ has a left derivative at 0 (possibly equal to $-\infty$) and the function ψ has a derivative at 0 because it is a rational function and 0 is not a pole. Then, we can write (here, the symbol $'$ has to be understood as a left derivative)

$$\phi(0) = \psi(0)\check{\psi}(0)$$

$$\phi'(0) = \check{\psi}'(0).\psi(0) + \check{\psi}(0)\psi'(0).$$

Since $\psi(0) \in [0, +\infty[$, $\check{\psi}(0) \in [0, +\infty[$, $\check{\psi}'(0) \in [-\infty, 0[$, $\psi'(0) \in]0, +\infty]$ (that are obvious consequences of Lévy-Kinchin formula for exponent of subordinator or opposite of a subordinator), we obtain :

$$\phi(0) = 0 \Longrightarrow \left(\psi(0) = 0 \text{ and } \phi'(0) \geq 0\right) \text{ or } \left(\check{\psi}(0) = 0 \text{ and } \phi'(0) \leq 0\right)$$

And the proof is finished.   □

In the sequel we will normalize ψ by setting $\psi_\infty = 1$.
We will always denote
$$-\beta_1, -\beta_2, \ldots, -\beta_m$$
the (possibly equal) roots of ψ and
$$m_i = \sharp\{k < i; \beta_k = \beta_i\}$$
($m_i = 0$ for all i's iff the roots $-\beta_1, \ldots, -\beta_m$ of ψ are simple).
Remark If X is of bounded variations and has a non positive drift then $m = n$. In all other cases, we have $m = n + 1$, ψ is then the exponent of a subordinator with positive drift and X "creeps upwards" (see theorem 19 chapter 6 of [B96]).
Now, we introduce few notations which will be useful in the sequel, and which are related to the negative Wiener-Hopf factor $\check{\psi}$.
First, $\check{\psi}$ being the exponent of the opposite of a subordinator, the inverse of $\check{\psi}$, $\frac{1}{\check{\psi}}$, is the Laplace transform of a measure supported by $]-\infty, 0]$. We will denote this measure by $\check{U}(dy)$,

$$\frac{1}{\check{\psi}(\lambda)} =: \int_{]-\infty,0]} e^{-\lambda y}\check{U}(dy) \qquad (\Re(\lambda) < 0)$$

Also, for all $x \in]0, +\infty[$, we denote by $\check{U}_{[-x,0]}(dy)$ and $\check{U}_{]-\infty,-x[}(dy)$ the measures $1_{[-x,0]}(y)\check{U}(dy)$ and $1_{]-\infty,-x[}(y)\check{U}(dy)$ and by $\check{U}_{[-x,0]}(\lambda)$ and $\check{U}_{]-\infty,-x[}(\lambda)$ their Laplace transform;

$$\check{U}_{[-x,0]}(\lambda) := \int_{[-x,0]} e^{-\lambda y}\check{U}(dy) \quad (\lambda \in \mathbf{C})$$

$$\check{U}_{]-\infty,-x[}(\lambda) := \int_{]-\infty,-x[} e^{-\lambda y}\check{U}(dy) \qquad (\Re(\lambda) < 0)$$

Using standard facts about the relation between negative Wiener-Hopf factor $\check{\psi}$ and the fluctuations of the Lévy process (see chapter 6 of [B96]), we obtain the next proposition.

Proposition 2.4. *1) If $\phi(0) > 0$ or $\phi'(0) > 0$ then then the function $\frac{\check{\psi}(0)}{\check{\psi}(\lambda)}$ is the Laplace transform of the measure $\check{\psi}(0)\check{U}(dy)$ which is the distribution of m, the minimum of X ($m := \inf\{X_t; 0 \leq t < \zeta\}$).
2) $\check{\psi}(\lambda)\check{U}_{]-\infty,-x[}(\lambda)$ is the Laplace transform of the distribution $\mathbf{P}(X_{T_x} \in dy; T_x < \zeta)$, ($T_x := \inf\{t; X_t < -x\}$).*

Now, using the dual properties settled in this proposition which involves the positive Wiener-Hopf factor ψ instead of the negative one $\check{\psi}$ and using the explicit form of ψ given in theorem 2.3, we obtain, (see also Mordecki [LM08] theorem 2.2),

Corollary 2.5. 1) If $\phi(0) > 0$ or $\phi'(0) < 0$ then the distribution of the maximum,
$M := \{\sup; X_t; t \in [0, \zeta[\}$ is

$$\mathbf{P}(M \in dy) = a_0 \delta_0(dy) + \sum_{i=1}^{m} a_i \frac{y^{m_i}}{m_i!} e^{-\beta_i y} dy,$$

where the coefficients a_i are given by the rational expansion :

$$\frac{\Pi_{j=1}^{n}(1 + \lambda/\gamma_j)}{\Pi_{i=1}^{m}(1 + \lambda/\beta_i)} = a_0 + \sum_{i=1}^{m} \frac{a_i}{(\lambda + \beta_i)^{m_i+1}}$$

2) The distribution of the "over shoot" $X_{T^x} - x$ on $T^x < \zeta$ $(T^x = \inf\{t; X_t > x\})$
is the following

$$\mathbf{P}(X_{T^x} - x \in dy; T^x < \zeta) = c_0(x)\delta_0(dy) + \sum_{j=1}^{n} c_j(x)\frac{y^{n_j}}{n_j!} e^{-\gamma_j y} 1_{y>0} dy,$$

where the coefficients $c_j(x)$ are given by the rational expension :

$$c_0(x) + \sum_{j=1}^{n} \frac{c_j(x)}{(\lambda + \gamma_j)^{n_j+1}} = \frac{\Pi_{i=1}^{m}(1 + \lambda/\beta_i)}{\Pi_{j=1}^{n}(1 + \lambda/\gamma_j)} \cdot \sum_{i=1}^{m} \frac{a_i}{m_i!} \frac{\partial^{m_i}}{(-\partial\beta)^{m_i}} \left[\frac{e^{-\beta x}}{\lambda + \beta}\right]_{\beta=\beta_i}$$

Corollary 2.5 and Proposition 2.4 give us a foretaste of the explicit results that we
obtain in the sequel : Most of the explicit distributions that we will obtain will be of
the preceding form or of convolution of measures of that form; that is, a combination
of exponential or gamma distributions with explicit coefficients, possibly restricted
to an interval, and convoluted with (a possible restriction of) the measure $\check{U}(dy)$.
The next proposition introduces Laplace transforms of distributions that will be
involved in the sequel. The easy proof is left to the reader.

Proposition 2.6. *1)*

$$\mathbf{m}_i(x, \lambda) := \frac{\partial^{m_i}}{(-\partial\beta)^{m_i}} \left[\frac{e^{(\lambda+\beta)x}\check{U}_{[-x,0]}(-\beta) - \check{U}_{[-x,0]}(\lambda)}{\lambda + \beta}\right]_{\beta=\beta_i}$$

is the Laplace transform of the measure

$$\left((1_{y\in[0,x]}y^{m_i}e^{-\beta_i y}dy) * (1_{y\in[-x,0]}\check{U}(dy))\right)1_{y\in[-x,0]}$$

2)

$$\mathbf{n}_j(x, \lambda) := \frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}} \left[\frac{e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) - e^{\gamma x}\check{U}_{]-\infty,-x[}(-\gamma)}{\lambda + \gamma}\right]_{\gamma=\gamma_j}$$

is the Laplace transform of the measure

$$\left((y^{n_j}e^{-\gamma_j y}1_{y>0}) * (\check{U}(dy - x)1_{y<0})\right)1_{y<0}$$

3)

$$\mathbf{i}_j(\lambda) = \frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}} \left[\frac{1}{\lambda + \gamma}\right]_{\gamma=\gamma_j}$$

is the Laplace transform of the measure

$$y^{n_j}e^{-\gamma_j y}1_{y>0}dy$$

3. SOME MATRICES OF COEFFICIENTS

Recall first all of the notations already introduced.

$$-\gamma_1, \ldots, -\gamma_n$$

are the poles of ϕ (and ψ) located on the half plane $\{\lambda, \Re(\lambda) < 0\}$, and

$$n_j = \sharp\{k < j; \gamma_k = \gamma_j\}$$

$$-\beta_1, \ldots, -\beta_m$$

are the roots of ϕ (and ψ) located on the half plane $\{\lambda, \Re(\lambda) < 0\}$ together with 0 is $\phi(0) = 0$ and $\phi'(0) > 0$. and

$$m_i = \sharp\{k < i; \beta_k = \beta_i\}$$

We will denote by $\mathbf{l}$ and $\mathbf{c}$ the line and the column :

$$\mathbf{l} = (1_{n_j=0}; 1 \le j \le n).$$

$$\mathbf{c} = (1_{m_i=0}; 1 \le i \le m)^t.$$

The $\mathbf{m}_i(x, \lambda)$ and $\mathbf{n}_j(x, \lambda)$ and $\mathbf{i}_j(\lambda)$ being defined in the previous proposition 2.6, we denote by $\mathbf{m}(x, \lambda)$ the column

$$\mathbf{m}(x, \lambda) = (\mathbf{m}_i(x, \lambda); 1 \le i \le m)^t$$

and by $\mathbf{n}(x, \lambda)$ the line

$$\mathbf{n}(x, \lambda) = (\mathbf{n}_j(x, \lambda); 1 \le j \le n)$$

and by $\mathbf{i}(\lambda)$ the line

$$\mathbf{i}(\lambda) = (\mathbf{i}_j(\lambda); 1 \le j \le n)$$

Also, define the line $\mathbf{v}(x)$,

$$\mathbf{v}(x) := \left(-\left[\frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}} e^{\gamma x} \check{U}_{]-\infty, -x[}(-\gamma) \right]_{\gamma=\gamma_j}; 1 \le j \le n \right),$$

and $\mathbf{w}(x)$ be the column,

$$\mathbf{w}(x) := \left(\left[\frac{\partial^{m_i}}{(-\partial\beta)^{m_i}} e^{\beta_i x} \check{U}_{[-x, 0]}(-\beta) \right]_{\beta=\beta_i}; 1 \le i \le m \right)^t$$

Proposition 3.1. *If $m = n + 1$, then the limit*

$$\check{u}(-x) := \lim_{\lambda \to -\infty} (-\lambda) e^{-\lambda x} \check{U}_{]-\infty, -x[}(\lambda)$$

exists in $[0, +\infty[$ for all $x \in]0, +\infty[$.

This proposition will be proved incidently here as a consequence of our computations. In fact, it can be shown that the function $\check{u}$ is a density of the measure $1_{y<0}\check{U}(dy)$ and we will see that this density is of bounded variations (see remark 4 after corollary 5.2)

If $m = n + 1$, we will denote $\mathbf{w}'(x)$ the column $\left(\mathbf{w}'_i(x); 1 \le i \le m \right)^t$, with

$$\mathbf{w}'_i(x) := \frac{\partial^{m_i}}{(-\partial\beta)^{m_i}} \left[\beta e^{\beta x} \check{U}_{[-x, 0]}(\beta) + \check{u}(-x) \right]_{\beta=\beta_i}$$

Notice that if $\check{u}$ is continuous, $\mathbf{w}_i'(x)$ is the left derivative of $\mathbf{w}_i(x)$, hence the notation.

Let now $\mathbf{W}(x)$ be the matrix (m,n) :

$$\mathbf{W}(x) = (\mathbf{W}_{i,j}(x); 1 \le i \le m, 1 \le j \le n)$$

where,

$$\mathbf{W}_{i,j}(x) := \frac{\partial^{m_i}\partial^{n_j}}{(-\partial\beta)^{m_i}(-\partial\gamma)^{n_j}}\left[\frac{e^{\beta x}U_{[-x,0]}(-\beta) + e^{\gamma x}\check{U}_{]-\infty,-x[}(-\gamma)}{\beta - \gamma}\right]_{\beta=\beta_i,\gamma=\gamma_j}$$

When $m = n + 1$, denote $\tilde{\mathbf{W}}(x)$ the square matrix

$$\tilde{\mathbf{W}}(x) = \left[\begin{array}{cc} \mathbf{w}(x) & \mathbf{W}(x) \end{array}\right]$$

We will show later the next proposition

Let $\mathbf{W}$ be any matrix and let $\mathbf{f}$ be any line, the dimension of $\mathbf{f}$ being equal to the number of columns of $\mathbf{W}$, $\mathbf{W}_i^{\mathbf{f}}$ will be the matrix obtained from $\mathbf{W}$ by replacing the i-th line by $\mathbf{f}$. And $\mathbf{W}_i$ will denote the matrix obtained from $\mathbf{W}$ by just taking it's i-th line off.

Similarly, let $\mathbf{e}$ be a column whose dimension is equal to the number of lines of W, $\mathbf{W}_{\mathbf{e}}^j$ will refer to the matrix obtained from $\mathbf{W}$ by substituting $\mathbf{e}$ to it's j-th column.

4. THE DISTRIBUTION OF $(S_\zeta, X_\zeta, I_\zeta)$

Theorem 4.1. *The potential kernel of the triple (S_t, X_t, I_t) is characterized by the following identities. For all $\lambda_1, \lambda_2 \in \mathbf{C}$, $x \in]0, +\infty[$,*

$$\mathbf{E}(\int_0^\zeta e^{-\lambda_1 I_t}1_{S_t - I_t \le x}e^{-\lambda_2(X_t - I_t)}dt) = \mathbf{E}(\int_0^\zeta e^{-\lambda_2 S_t}1_{S_t - I_t \le x}e^{-\lambda_1(X_t - S_t)}dt)$$

$$= \frac{\left|\begin{array}{cc} \check{U}_{[-x,0]}(\lambda_1) & \mathbf{v}(x) \\ \mathbf{m}(x,\lambda_1) & \mathbf{W}(x) \end{array}\right|}{|\mathbf{W}(x)|}\frac{\left|\begin{array}{cc} 1 & \mathbf{l} \\ e^{-\lambda_2 x}\mathbf{m}(x,\lambda_2) & \mathbf{W}(x) \end{array}\right|}{|\mathbf{W}(x)|} \quad if \quad m = n$$

$$= \frac{\left|\begin{array}{ccc} \check{U}_{[-x,0]}(\lambda_1) & \check{u}(-x) & \mathbf{v}(x) \\ \mathbf{m}(x,\lambda_1) & \mathbf{w}(x) & \mathbf{W}(x) \end{array}\right|}{|\tilde{\mathbf{W}}(x)|}\frac{\left|\begin{array}{cc} e^{-\lambda_2 x}\mathbf{m}(x,\lambda_2) & \mathbf{W}(x) \end{array}\right|}{|\tilde{\mathbf{W}}(x)|} \quad if \quad m = n+1$$

Notations. Let, if $m = n$,

$$\check{a}(x) := \left|\begin{array}{cc} \check{U}_{[-x,0]}(0) & \mathbf{v}(x) \\ \mathbf{m}(x,0) & \mathbf{W}(x) \end{array}\right| \qquad a(x) := \left|\begin{array}{cc} 1 & \mathbf{l} \\ \mathbf{m}(x,0) & \mathbf{W}(x) \end{array}\right| \qquad r(x) := |\mathbf{W}(x)|$$

And, if $m = n+1$,

$$\check{a}(x) := \left|\begin{array}{ccc} \check{U}_{[-x,0]}(0) & \check{u}(-x) & \mathbf{v}(x) \\ \mathbf{m}(x,0) & \mathbf{w}(x) & \mathbf{W}(x) \end{array}\right| \qquad a(x) := \left|\begin{array}{cc} \mathbf{m}(x,0) & \mathbf{W}(x) \end{array}\right| \qquad r(x) := |\tilde{\mathbf{W}}(x)|$$

Corollary 4.2. *If $\phi(0) > 0$, then the law of the triple $(S_\zeta, X_\zeta, I_\zeta)$ is characterized as follow :*

$$\frac{1}{\phi(0)}\mathbf{P}(S_\zeta - I_\zeta \le x) = \frac{\check{a}(x)a(x)}{r^2(x)}.$$

Given $S_\zeta - I_\zeta \leq x$, the random variable $X_\zeta - I_\zeta$ is independent of I_ζ and has the same distribution as S_ζ and

$$\mathbf{P}(I_\zeta \in dy | S_\zeta - I_\zeta \leq x) = \frac{1_{y\in[-x,0]}}{\check{a}(x)}\left(\check{a}_0\check{U}(dy) - [1_{y\in[-x,0]}\check{U}(dy)]*[\sum_{i=1}^m \check{a}_i(x)y^{m_i}e^{-\beta_i y}1_{y\in[0,x]}dy]\right)$$

$$\mathbf{P}(S_\zeta \in dy | S_\zeta - I_\zeta \leq x) = \frac{1_{y\in[0,x]}}{a(x)}\left(a_0\delta_0(dy) - [1_{y\in[0,x]}\check{U}(dy-x)]*[\sum_{i=1}^m a_i(x)y^{m_i}e^{-\beta_i y}1_{y\in[0,x]}dy]\right)$$

with, if $m = n$,

$$\check{a}_0 = a_0 = 1$$

$$\check{a}_i(x) = \frac{|\mathbf{W}_i^{\mathbf{v}(x)}(x)|}{|\mathbf{W}(x)|} \qquad a_i(x) = \frac{|\mathbf{W}_i^{\mathbf{1}}(x)|}{|\mathbf{W}(x)|}, \qquad i = 1\ldots m$$

And, if $m = n + 1$,

$$\check{a}_0 = 1 \qquad a_0 = 0$$

$$\check{a}_i(x) = \frac{|\tilde{\mathbf{W}}(x)_i^{[\check{u}(-x),\mathbf{v}(x)]}(x)|}{|\tilde{\mathbf{W}}(x)||} \qquad a_i(x) = -\frac{|\mathbf{W}_i(x)|}{|\tilde{\mathbf{W}}(x)|} \qquad i = 1\ldots m$$

The identities are still true when $\phi(0) = 0$ and when $\dfrac{1}{\phi(0)}.\mathbf{P}(S_\zeta - I_\zeta \leq x)$,

$$\mathbf{P}(S_\zeta \in dy | S_\zeta - I_\zeta \leq x) \qquad \mathbf{P}(I_\zeta \in dy | S_\zeta - I_\zeta \leq x)$$

are replaced by $\displaystyle\int_0^{+\infty} \mathbf{P}(S_t - I_t \leq x)dt$,

$$\frac{\int_0^{+\infty} \mathbf{P}(S_t - I_t \leq x, S_t \in dy))dt}{\int_0^{+\infty} \mathbf{P}(S_t - I_t \leq x)dt} \qquad \frac{\int_0^{+\infty} \mathbf{P}(S_t - I_t \leq x, I_t \in dy)dt}{\int_0^{+\infty} \mathbf{P}(S_t - I_t \leq x)dt}.$$

Remark Let ϕ_0 be a proper (not killed) Lévy exponent and apply the formula of theorem to $\phi = \phi_0 + q$ when q vary from 0 to $+\infty$, let $\mathbf{P}_q$ be the corresponding distribution, that is the distribution of the Lévy process X, with Lévy exponent ϕ_0, killed at an independent exponential time of rate q. Clearly we have

$$\mathbf{E}_q\Big(\int_0^\zeta e^{-\lambda_1 I_t}1_{S_t - I_t \leq x}e^{-\lambda_2(X_t - I_t)}dt\Big) = \mathbf{E}_0\Big(\int_0^{+\infty} e^{-\lambda_1 I_t}1_{S_t - I_t \leq x}e^{-\lambda_2(X_t - I_t)}e^{-qt}dt\Big)$$

When taking the inverse Laplace transform over q, λ_1 and λ_2, we find the measure

$$\mathbf{P}_0(I_t \in dy, X_t - I_t \in dz, S_t - I_t \leq x)dt$$

Then the previous theorem and corollary give a characterization of the distribution of (S_T, X_T, I_T) at any time T.

As an application, one can compute the price of a double barrier knock-out call option under exponential Lévy model for the asset Y, $Y_t = Y_0 e^{X_t}$, when the underlying Lévy process X satisfy the assumption 2.1, that is, an european call option with strike K which is desactivated if the asset Y goes down a barrier value a or up a value b before the time of maturity T. It is clear that the price is given by the expression :

$$\mathbf{E}_0\big((Y_0 e^{X_T} - K)1_{X_T > \log\frac{K}{Y_0}}1_{I_T > \log\frac{a}{Y_0}}1_{S_T < \log\frac{a}{Y_0}}\big)$$

Then, this price can be deduced from the distribution of the triple (S_T, X_T, I_T).

5. BEHAVIOR OF THE PROCESS (X, S) AT TIME $V_x = \inf\{t, X_t - S_t < -x\}$

Theorem 5.1.

$$
\mathbf{E}\left(\exp\left(-\mu_2 S_{V_x} - \mu_1(X_{V_x} - S_{V_x})\right)1_{V_x < \zeta}\right) = \check{\psi}(\mu_1)\frac{\begin{vmatrix} \check{U}_{]-\infty,-x[}(\mu_1) & e^{\mu_1 x}.\mathbf{n}(x,\mu_1) \\ \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{\begin{vmatrix} 1 & \mathbf{i}(\mu_2) \\ \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}} \quad if \quad m = n
$$

and

$$
= \check{\psi}(\mu_1)\frac{\begin{vmatrix} \mu_1\check{U}_{]-\infty,-x[}(\mu_1) & -\check{U}_{]-\infty,-x[}(\mu_1) & e^{\mu_1 x}.\mathbf{n}(x,\mu_1) \\ \mathbf{w}'(x) & \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{\begin{vmatrix} \mu_2 & -1 & \mathbf{i}(\mu_2) \\ \mathbf{w}'(x) & \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}} \quad if \quad m = n+1
$$

Notations. Denote if $m = n$,

$$
b(x) := \begin{vmatrix} 1 & -\mathbf{i}(0) \\ \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix} \qquad \check{c}(x) := \lim_{\lambda \to 0} \check{\psi}(\lambda) \begin{vmatrix} \check{U}_{]-\infty,-x[}(\lambda) & -\mathbf{n}(x,\lambda) \\ \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}.
$$

And, if $m = n+1$,

$$
b(x) := \begin{vmatrix} 0 & -1 & \mathbf{i}(0) \\ \mathbf{w}'(x) & \mathbf{w}(x) & W(x) \end{vmatrix} \qquad \check{c}(x) := \lim_{\lambda \to 0} \check{\psi}(\lambda) \begin{vmatrix} \lambda\check{U}_{]-\infty,-x[}(\lambda) & -\check{U}_{]-\infty,-x[}(\lambda) & \mathbf{n}(x,\lambda) \\ \mathbf{w}'(x) & \mathbf{w}(x) & W(x) \end{vmatrix}
$$

Corollary 5.2.

$$
\mathbf{P}(V_x < \zeta) = \frac{\check{c}(x)}{b(x)},
$$

The random variables S_{V_x} and $X_{V_x} - S_{V_x}$ are independent conditionally to $V_x < \zeta$ and the distribution of $X_{V_x} - S_{V_x}$ on $V_x < \zeta$ is characterized by the identity,

$$
\check{U}(dz) * \mathbf{P}(X_{V_x} - S_{V_x} \in dz; V_x < \zeta)
$$

$$
= [b(x)]^{-1}.\left[1_{z<-x}\check{U}(dz)\right] * \left[\check{c}_{-1}\delta_0'(dz) + \check{c}_0(x)\delta_0(dz) - \sum_1^n \check{c}_j(x)z^{k_j}e^{-\gamma_j z}1_{z<-x}(dz)\right]
$$

And,

$$
\mathbf{P}(S_{V_x} \in dy; V_x < \zeta) = \check{c}(x)\left(b_0\delta_0(dy) + \sum_{i=1}^m b_i(x)y^{m_i(x)}e^{-\beta_i(x)y}1_{y>0}dy\right).
$$

Where, if $m = n$,

$$
\check{c}_{-1} = 0 \qquad \check{c}_0(x) = -1 \qquad \check{c}_j(x) = -\left|\mathbf{W}(x)_{[\mathbf{w}(x)]}^j\right| \qquad j = 0, \ldots, n
$$

And if $m = n+1$, then

$$
\check{c}_{-1} = 1 \qquad \check{c}_j(x) = \left|\tilde{\mathbf{W}}(x)_{[\mathbf{w}'(x)]}^{j+1}\right| \qquad j = 0, \ldots, n
$$

The $\beta_i(x)$, the exponents $m_i(x) := \sharp\{k < i; \beta_k(x) = \beta_i(x)\}$ and the coefficients $b_i(x)$ are given by the following fractional expansion

$$
b_0(x) + \sum_{i=1}^m \frac{b_i(x)}{(\lambda + \beta_i(x))^{m_i(x)+1}} = [\check{c}_{-1}(x)\lambda + \check{c}_0(x) - \sum_{j=1}^n \check{c}_j(x)\frac{1}{(\lambda + \gamma_j)^{n_j+1}}]^{-1}
$$

$$= \left| \begin{array}{cc} 1 & -\mathbf{i}(\lambda) \\ 1 & \mathbf{W}(x) \end{array} \right|^{-1} \quad \text{if} \quad m = n$$

$$= \left| \begin{array}{ccc} \lambda & -1 & \mathbf{i}(\lambda) \\ \mathbf{w}'(x) & \mathbf{w}(x) & \mathbf{W}(x) \end{array} \right|^{-1} \quad \text{if} \quad m = n+1$$

Remark

1) Notice that, when $\check{\psi}(0) = 0$, that means that X does not drift to $+\infty$ (see lemma 2, chapter 6 of [B96]) then

$$\mathbf{P}(X_{T_x} < \zeta) = \mathbf{P}(X_{T_x} < +\infty) = 1,$$

thus

$$\check{\psi}(\lambda)\check{U}_{]-\infty,-x[}(\lambda) = \mathbf{E}(e^{-\lambda X_{T_x}}; T_x < +\infty) \to 1 \qquad \lambda \to 0$$

and we get

$$\lim_{\lambda \to 0} \check{\psi}(\lambda)n_j(x,\lambda) = \lim_{\lambda \to 0} \frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}} \left[\frac{\check{\psi}(\lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) - \check{\psi}(\lambda)e^{\gamma x}\check{U}_{]-\infty,-x[}(-\gamma)}{\lambda + \gamma} \right]_{\gamma = \gamma_j}$$

$$= \frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}} \left[\frac{1}{\gamma} \right]_{\gamma = \gamma_j} = \mathbf{i}_j(0)$$

Thus, we see that $b(x) = \check{c}(x)$ when $\check{\psi}(0) = 0$ and, according to the previous corollary, we obtain that $\mathbf{P}(V_x < +\infty) = 1$ when X does not drift to $+\infty$: this means that the heights of the excursions out of the set $\{t, X_t = S_t\}$ are unbounded a.s. This can be obtained by elementary trajectorial arguments.

2) In order to obtain a direct characterization of the distribution $\mathbf{P}(X_{V_x} - S_{V_x} \in dz; V_x < \zeta)$ instead of the characterization of the convoluted distribution $\check{U}(dy) * \mathbf{P}(X_{V_x} - S_{V_x} \in dz; V_x < \zeta)$, it is necessary to introduce the drift, the Lévy measure and the killing rate associated to the exponent $\check{\psi}$. We leave this part to the interested reader.

3) Due to the lake of memory of exponential distributions and the fact that

$$X_{V_x} 1_{X_{V_x} > 0} = (S_{V_x} + (X_{V_x} - S_{V_x}))^+,$$

the distribution of X_{V_x} given $X_{V_x} > 0$ is, like the distribution of S_{V_x} as stated in the preceding corollary, a linear combination of the functions $y^{m_i(x)}e^{-\beta_i(x)y}1_{y>0}$.

4) It is clear from the characterisation of the distribution of $X_{V_x} - S_{V_x}$ when $m = n+1$ that the convolution $1_{z<-x}\check{U}(dz) * \delta'(dz)$ is a signed measure, that means that the measure $\check{U}(dy)$ has a density, and this density, which is the fonction $\check{u}$ appearing in lemma 3.1, is of bounded variations.

6. BEHAVIOR OF THE PROCESS (I, X) AT TIME $V^x := \inf\{t; X_t - I_t > x\}$

Theorem 6.1.

$$\mathbf{E}(e^{-\mu_1 I_{V^x}} e^{-\mu_2(X_{V^x} - I_{V^x})}) = \frac{e^{-\mu_2 x}}{\check{\psi}(\mu_1)} \frac{\left| \begin{array}{cc} 0 & \mathbf{i}(\mu_2) \\ \mathbf{c} & W(x) \end{array} \right|}{\left| \begin{array}{cc} 1 & \mathbf{n}(x,\mu_1) \\ \mathbf{c} & W(x) \end{array} \right|} \quad \text{if} \quad m = n$$

$$= \frac{e^{-\mu_2 x}}{\check{\psi}(\mu_1)} \cdot \frac{\left| \begin{array}{ccc} 0 & 1 & \mathbf{i}(\mu_2) \\ \mathbf{c} & \mathbf{w}(x) & \mathbf{W}(x) \end{array} \right|}{\left| \begin{array}{ccc} 1 & e^{-\mu_1 x}\check{U}_{]-\infty,-x[}(\mu_1) & \mathbf{n}(x,\mu_1) \\ \mathbf{c} & \mathbf{w}(x) & \mathbf{W}(x) \end{array} \right|} \quad \text{if} \quad m = n+1$$

Notations If $m = n$,

$$c(x) = \begin{vmatrix} 0 & \mathbf{i}(0) \\ \mathbf{c} & \mathbf{W}(x) \end{vmatrix} \qquad \check{b}(x) = \lim_{\lambda \to 0} \check{\psi}(\lambda) \begin{vmatrix} 1 & \mathbf{n}(x,\lambda) \\ \mathbf{c} & \mathbf{W}(x) \end{vmatrix}$$

If $m = n + 1$,

$$c(x) = \begin{vmatrix} 0 & 1 & \mathbf{i}(0) \\ \mathbf{c} & \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix} \qquad \check{b}(x) = \lim_{\lambda \to 0} \check{\psi}(\lambda) \begin{vmatrix} 1 & e^{-\lambda x}\check{U}_{]-\infty,-x[}(-\lambda) & \mathbf{n}(x,\lambda) \\ \mathbf{c} & \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}$$

One can notice again that $\check{b}(x) = c(x)$ when $\check{\psi}(0) = 0$.

Corollary 6.2.

$$\mathbf{P}(V^x < \zeta) = \frac{c(x)}{b(x)}$$

and I_{V^x} and $X_{V^x} - I_{V^x}$ are independent conditionally to $V^x < \zeta$

$$\mathbf{P}(X_{V^x} - I_{V^x} \in x + dy | V^x < \zeta) = [c(x)]^{-1} \cdot \left(c_0(x)\delta_0(dy) + \sum_{j=1}^{n} c_j(x) y^{n_j} e^{-\gamma_j y} 1_{y>0} dy \right)$$

$$\mathbf{P}(I_{V^x} \in dz; V_x < \zeta) = \frac{c(x)}{r(x)} \left[\check{U}(dz) \right] * \left[\sum_{n=0}^{+\infty} \mu_x^{*n}(dz) \right]$$

with

$$\mu_x(dz) = \left[1_{z<0} \cdot \check{U}(dz - x) \right] * \left[c_0(x)\delta_0(dz) + \sum_{j=1}^{n} c_j(x) z^{n_j} e^{-\gamma_j z} 1_{z>0} dz \right] 1_{z<0},$$

and, if $m = n$,

$$c_0(x) = 0 \quad \text{and} \quad c_j(x) = \frac{|W_{\mathbf{c}}^j(x)|}{|\mathbf{W}(x)|} \quad \text{for} \quad j = 1, \dots, n$$

If $m = n + 1$,

$$c_j(x) = \frac{|\tilde{\mathbf{W}}_{\mathbf{c}}^{j+1}(x)|}{|\tilde{\mathbf{W}}(x)|} \quad \text{for} \quad j = 0, \dots, n$$

7. BEHAVIOR OF THE PROCESS AT TIME $U_x = \inf\{t, S_t - I_t > x\}$.

Theorem 7.1.

$$\mathbf{P}(U_x < \zeta, X_{U_x} = I_{U_x}) = \frac{a(x)\check{c}(x)}{r(x)^2} \qquad \mathbf{P}(U_x < \zeta, X_{U_x} = S_{U_x}) = \frac{\check{a}(x)c(x)}{r(x)^2}$$

Given the event $\{U_x < \zeta; X_{U_x} = I_{U_x}\}$, S_{U_x} and $X_{U_x} - S_{U_x}$ are independent. The random variable S_{U_x} has the same distribution as S_ζ given $S_\zeta - I_\zeta \le x$ if $\phi(0) > 0$, and has the distribution $\dfrac{\int_0^\infty \mathbf{P}(S_t \in dz, S_t - I_t \le x) dt}{\int_0^\infty \mathbf{P}(S_t - I_t \le x) dt}$ if $\phi(0) = 0$. The random variable $X_{U_x} - S_{U_x}$ has the same distribution as $X_{V_x} - S_{V_x}$ given $V_x < \zeta$.

Given the event $\{U_x < \zeta; X_{U_x} = S_{U_x}\}$, I_{U_x} and $X_{U_x} - I_{U_x}$ are independent. The random variable I_{U_x} has the same distribution as I_ζ given $S_\zeta - I_\zeta \le x$ if $\phi(0) > 0$, and has the distribution $\dfrac{\int_0^\infty \mathbf{P}(I_t \in dy, S_t - I_t \le x) dt}{\int_0^\infty \mathbf{P}(S_t - I_t \le x) dt}$ if $\phi(0) = 0$. The random variable $X_{U_x} - I_{U_x}$ has the same distribution as $X_{V^x} - I_{V^x}$ given $V^x < \zeta$.

8. Behavior of the process at time $T_a^b = \inf\{t; X_t \notin [-a,b]\}$

Theorem 8.1. *We have, if $\phi(0) > 0$*

$$\mathbf{P}(X_{T_a^b} = S_{T_a^b}, T_a^b < \zeta) = \int_{[-a,0]} \frac{c(b-y)}{a(b-y)} \frac{\mathbf{P}(I_\zeta \in dy, S_\zeta \leq b)}{\phi(0)}.$$

And

$$\mathbf{P}(X_{T_a^b} = I_{T_a^b}, T_a^b < \zeta) = \int_{[0,b]} \frac{\check{c}(a+x)}{\check{a}(a+x)} \frac{\mathbf{P}(I_\zeta \geq -a, S_\zeta \in dx)}{\phi(0)}$$

If $\phi(0) = 0$, we replace in last formulas $\frac{\mathbf{P}(I_\zeta \in dy, S_\zeta \leq b)}{\phi(0)}$ and $\frac{\mathbf{P}(I_\zeta \geq -a, S_\zeta \in dx)}{\phi(0)}$ by respectively

$$\int_0^{+\infty} \mathbf{P}(I_t \in dy, S_t \leq b)dt \quad and \quad \int_0^{+\infty} \mathbf{P}(S_t \in dx, -I_t \geq -a)dt$$

Given $\{X_{T_a^b} = S_{T_a^b}, T_a^b < \zeta\}$, the distribution of $I_{T_a^b}$ is the distribution of I_ζ given $\{S_\zeta \leq b, I_\zeta \geq -a\}$ if $\phi(0) > 0$ and the distribution $\dfrac{(\int_0^{+\infty} \mathbf{P}(I_t \in dy, S_t \leq b)dt}{\int_0^{+\infty} \mathbf{P}(S_t \leq b, -I_t \geq -a)dt}$ if $\phi(0) = 0$. The distribution of $X_{T_a^b} - I_{T_a^b}$ given $I_{T_a^b} = y$ is the one of $X_{V^{b+y}}$ given $V^{b+y} < \zeta$.
Given $\{X_{T_a^b} = I_{T_a^b}, T_a^b < \zeta\}$, the distribution of $S_{T_a^b}$ is the distribution of S_ζ given $\{S_\zeta \geq -a, I_\zeta \leq b\}$ if $\phi(0) > 0$ and the distribution $\dfrac{\int_0^{+\infty} \mathbf{P}(S_t \in dx, -I_t \geq -a)dt}{\int_0^{+\infty} \mathbf{P}(S_t \leq b; -I_t \geq -a)dt}$ if $\phi(0) = 0$. The distribution of $X_{T_a^b} - S_{T_a^b}$ given $S_{T_a^b} = x$ is the one of $X_{V_{a+x}}$ given $V_{a+x} < \zeta$.

Remarks 1) We can deduce from this theorem that the distribution of $X_{T_a^b} - b$ given $X_{T_a^b} > b$ is a linear combination of distribution of the distributions $y^{n_j} e^{-\gamma_j y} 1_{y>0} dy$. That could easily be deduced from the lack of memory of the exponential distribution. The previous result allows also to compute a closed form of the coefficients.
2) As in the remark following theorem 4.1, let ϕ_0 be a proper (not killed) Lévy exponent and apply the formula of theorem to $\phi = \phi_0 + q$ when q vary from 0 to $+\infty$, let $\mathbf{P} = \mathbf{P}_q$ be the corresponding distribution, that is the distribution of the Lévy process X, with Lévy exponent ϕ_0, killed at an independent exponential time of rate q. The probabilities $\mathbf{P}_q(X_{T_a^b} = S_{T_a^b}, T_a^b < \zeta)$ and $\mathbf{P}_q(X_{T_a^b} = I_{T_a^b}, T_a^b < \zeta)$ are q times the Laplace transforms of the functions of t,

$$\mathbf{P}_0(X_{T_a^b} = S_{T_a^b}, T_a^b < t) \quad and \quad \mathbf{P}_0(X_{T_a^b} = S_{T_a^b}, T_a^b < t).$$

On the other hand, the price of a double barrier with rebate, say rebate r^+ (resp. r^-) if the option exceed the value b (resp. goes down the value a) before time T. (Denote $\tilde{a} = -\log\frac{a}{Y_0}$, $\tilde{b} = \log\frac{b}{Y_0}$) is given by the next formula

$$\mathbf{E}((Y_0 e^{X_T} - K) 1_{X_T > \log \frac{K}{Y_0}} 1_{I_T > -\tilde{a}; S_T < \tilde{b}}) + r^+ \mathbf{P}(X_{T_{\tilde{a}}^{\tilde{b}}} = S_{T_{\tilde{a}}^{\tilde{b}}}, T_{\tilde{a}}^{\tilde{b}} < T)$$

$$+ r^- \mathbf{P}(X_{T_{\tilde{a}}^{\tilde{b}}} = I_{T_{\tilde{a}}^{\tilde{b}}}, T_{\tilde{a}}^{\tilde{b}} < T)$$

We have seen that the first term can be computed with the help of theorem 4.1 and now we see that the two other terms can be computed by an inverse Laplace

transform in time of $\mathbf{P}_q(X_{T_{\tilde{a}}^{\tilde{b}}} = S_{T_{\tilde{a}}^{\tilde{b}}}, T_{\tilde{a}}^{\tilde{b}} < \zeta)$ and $\mathbf{P}_q(X_{T_{\tilde{a}}^{\tilde{b}}} = I_{T_{\tilde{a}}^{\tilde{b}}}, T_{\tilde{a}}^{\tilde{b}} < \zeta)$ which are characterized in the preceding theorem.

9. RECALL OF THE MAIN RESULT OF [F10] AND ITS APPLICATION WITH THE ASSUMPTION 2.1

Proposition 9.1. *There exist unique functions $A(x,\lambda)$, $\check{A}(x,\lambda)$, $B(x,\lambda)$ and $\check{B}(x,\lambda)$, $C(x,\lambda)$ and $\check{C}(x,\lambda)$ such that $A(x,\lambda)$ and $\check{A}(x,\lambda)$, are defined for $\lambda \in \mathbf{C}$ and entire; $B(x,\lambda)$ and $C(x,\lambda)$ are defined, continuous on $\{\Re(\lambda) \geq 0\}$ and holomorphic on $\{\Re(\lambda) > 0\}$; $\check{B}(x,\lambda)$ and $\check{C}(x,\lambda)$ are defined and continuous on $\{\Re(\lambda) \leq 0\}$, and holomorphic on $\{\Re(\lambda) < 0\}$, satisfying the following identities,*

$$\mathbf{P}\Big(\int_0^\zeta e^{-\lambda_1 I_t} 1_{S_t - I_t \leq x} e^{-\lambda_2(X_t - I_t)} dt\Big) = \mathbf{P}\Big(\int_0^\zeta e^{-\lambda_2 S_t} 1_{S_t - I_t \leq x} e^{-\lambda_1(X_t - S_t)} dt\Big)$$

$$= \check{A}(x,\lambda_1) A(x,\lambda_2)$$

$$\mathbf{E}\Big(\exp\big(-\mu_2 S_{V_x} - \mu_1(X_{V_x} - S_{V_x}) 1_{V_x < \zeta}\big)\Big) = \frac{\check{C}(x,\mu_1)}{B(x,\mu_2)}$$

$$\mathbf{E}\Big(\exp\big(-\mu_1 I_{V^x} - \mu_2(X_{V^x} - I_{V^x}) 1_{V^x < \zeta}\big)\Big) = \frac{C(x,\mu_2)}{\check{B}(x,\mu_1)}$$

$$\psi(\lambda) A(x,\lambda) \to 1 \qquad B(x,\lambda) \sim \psi(\lambda) \quad for \quad \lambda \to +\infty$$

$$\check{B}(x,\lambda) \sim \check{\psi}(\lambda) \quad for \quad \lambda \to -\infty$$

Proof. The existence of the 6 functions $A, \check{A}, B, \check{B}, C, \check{C}$ has been settled in the general setting of a Lévy process in proposition 6.1 and proposition 6.2 of [F10] , the behavior of the functions is given in theorem 4.2 of [F10]. The uniqueness is obvious.

$\square$

More over, we know from theorem 4.2 of [F10] (or we can deduce them from proposition 9.1), the behavior of the six functions as settled in next proposition

Proposition 9.2. *The functions of λ, $A(x,\lambda)$, $e^{-\lambda x}\check{A}(x,\lambda)$, $\frac{\check{B}(x,\lambda)}{|\lambda|+1}$, $e^{\lambda x}C(x,\lambda)$ are bounded on $\{Re(\lambda) \geq 0\}$.*
The functions $\check{A}(x,\lambda)$, $e^{\lambda x}A(x,\lambda)$, $\frac{\check{B}(x,\lambda)}{|\lambda|+1}$, $e^{-\lambda x}\check{C}(x,\lambda)$ are bounded on $\{Re(\lambda) \leq 0\}$.
For $\Re(\lambda) \to +\infty$ $e^{\lambda x}C(x,\lambda) \to 0$ if $\psi(\lambda)$ is bounded, and for $\Re(\lambda) \to -\infty$, $e^{-\lambda x}\check{C}(x,\lambda) \to 0$ if $\check{\psi}(\lambda)$ is bounded.

Remark The property "$\psi(\lambda)$ is bounded" means that ψ is the exponent of a compound Poisson process and this is equivalent to the property (see chapter 6 of [B96] that the first time the Lévy process X visits $]0,+\infty[$ is non zero a.s. (here it is also equivalent to the condition $m = n$).
In the next lemma, we state that the determination of these six functions reduces to the computation of four polynomials.

Proposition 9.3. *There exist unique polynomials of λ, $P_1(x,\lambda)$, $Q_1(x,\lambda)$, $P_2(x,\lambda)$, $Q_2(x,\lambda)$, with*

$$\deg P_1 = n \quad \deg P_2 = m \quad \deg Q_1 \le m-1 \quad \deg Q_2 \le m-1,$$

such that

$$A(x,\lambda) = \frac{P_1(x,\lambda) + Q_1(x,\lambda)e^{-\lambda x}\check{U}_{[-x,0]}(\lambda)}{\Pi_1^m(\lambda+\beta_i)}$$

$$\check{A}(x,\lambda) = \frac{Q_2(x,\lambda)e^{\lambda x} + P_2(x,\lambda)\check{U}_{[-x,0]}(\lambda)}{\Pi_1^m(\lambda+\beta_i)}$$

$$B(x,\lambda) = \frac{P_2(x,\lambda)}{\Pi_1^n(\lambda+\gamma_j)}$$

$$\check{B}(x,\lambda) = \check{\psi}(\lambda)\frac{(P_1(x,\lambda) - Q_1(x,\lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda))}{\Pi_1^n(\lambda+\gamma_j)}$$

$$C(x,\lambda) = \frac{e^{-\lambda x}Q_1(x,\lambda)}{\Pi_1^n(\lambda+\gamma_j)}$$

$$\check{C}(x,\lambda) = \check{\psi}(\lambda).\frac{(P_2(x,\lambda)\check{U}_{]-\infty,-x[}(\lambda) - Q_2(x,\lambda)e^{\lambda x})}{\Pi_1^n(\lambda+\gamma_j)}$$

Proof. of proposition 9.3 , proposition 3.1, We have seen in [F10] theorem 4.2 that the matrix $M(x,\lambda)$ defined as follows

$$M(x,\lambda) := \begin{pmatrix} A(x,\lambda) & -C(x,\lambda) \\ \check{A}(x,\lambda) & B(x,\lambda) \end{pmatrix} \text{ if } \quad \Re(\lambda) > 0,$$

$$M(x,\lambda) := \begin{pmatrix} \check{B}(x,\lambda) & A(x,\lambda) \\ -\check{C}(x,\lambda) & \check{A}(x,\lambda) \end{pmatrix} \text{ if } \quad \Re(\lambda) < 0.$$

satisfies the identity

$$(9.1) \qquad\qquad M^+(x,iu) = M^-(x,iu)\begin{pmatrix} 0 & -1 \\ 1 & \phi(iu) \end{pmatrix}$$

Moreover, $\det M(x,\lambda) = 1$, and M is invertible. On the other hand, one can easily check that the matrix

$$N(x,\lambda) = \begin{pmatrix} 1 & 0 \\ \check{U}_{[-x,0]}(\lambda) & 1 \end{pmatrix}\begin{pmatrix} \Pi_1^m(\lambda+\gamma_j) & 0 \\ 0 & \Pi_1^m(\lambda+\beta_i) \end{pmatrix} \text{for} \quad \Re(\lambda) > 0$$

$$N(x,\lambda) = \begin{pmatrix} \check{\psi}(\lambda) & 1 \\ -\psi(\lambda).\check{U}_{]-\infty,-x[}(\lambda) & \check{U}_{[-x,0]}(\lambda) \end{pmatrix}\begin{pmatrix} \Pi_1^m(\lambda+\beta_i) & 0 \\ 0 & \Pi_1^n(\lambda+\gamma_j) \end{pmatrix} \text{for} \quad \Re(\lambda) < 0$$

satisfies the same identity 9.1.

Thus, the product $N(x,\lambda)M^{-1}(x,\lambda)$ is entire. One can see easily that, because of the boundary conditions fulfilled by the components of $M(x,\lambda)$ given in proposition 9.2, that this matrix $N(x,\lambda)M^{-1}(x,\lambda)$ is of the form :

$$N(x,\lambda)M^{-1}(x,\lambda) = \begin{pmatrix} P_2(x,\lambda) & -e^{-\lambda x}Q_1(x,\lambda) \\ -e^{\lambda x}Q_2(x,\lambda) & P_1(x,\lambda) \end{pmatrix} =: R(x,\lambda)$$

Where P_1, Q_1, P_2 and Q_2 are polynomials of λ. Then

$$\det R = \det N = \Pi_1^m(\lambda+\beta_i)\Pi_1^n(\lambda+\gamma_j),$$

and we obtain

$$M(x, \lambda) = R^{-1}(x, \lambda).N(x, \lambda)$$

$$= \begin{pmatrix} P_1(x, \lambda) & e^{-\lambda x}Q_1(x, \lambda) \\ e^{\lambda x}Q_2(x, \lambda) & P_2(x, \lambda) \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ \check{U}_{[-x,0]}(\lambda) & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\Pi_1^m(\lambda+\beta_i)} & 0 \\ 0 & \frac{1}{\Pi_1^n(\lambda+\gamma_j)} \end{pmatrix} \text{ for } \Re(\lambda) > 0$$

$$= \begin{pmatrix} P_1(x, \lambda) & e^{-\lambda x}Q_1(x, \lambda) \\ e^{\lambda x}Q_2(x, \lambda) & P_2(x, \lambda) \end{pmatrix} \times \begin{pmatrix} \check{\psi}(\lambda) & 1 \\ -\check{\psi}(\lambda).\check{U}_{]-\infty,-x[}(\lambda) & \check{U}_{[-x,0]}(\lambda) \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\Pi_1^n(\lambda+\gamma_j)} & 0 \\ 0 & \frac{1}{\Pi_1^m(\lambda+\beta_i)} \end{pmatrix} \text{ for } \Re(\lambda) < 0$$

Developping each term of this matrix product, we obtain the identities of the theorem. Now, according to proposition 9.2, $e^{-\lambda x}C(x, \lambda)$ is bounded on $\{\Re(\lambda) > 0\}$ and goes to 0 if ψ is bounded (that is if $m = n$), that gives that $\deg Q_1 \leq m - 1$. Futhermore, the equivalence of the proposition 9.1 $B(x, \lambda) \sim \psi(\lambda)$ gives that $P_2(x, \lambda) = \lambda^m$ for $\lambda \to +\infty$. Since $\psi(\lambda)A(x, \lambda) \to 1$, we deduce that $P_1(\lambda) \sim \lambda^n$ for $\lambda \to +\infty$.

Also $e^{-\lambda x}\check{C}(x, \lambda)$ is bounded for $\Re(\lambda) \to +\infty$ and goes to 0 if $\check{\psi}(\lambda)$ is bounded. This is equivalent to say that

$$\frac{P_2(x, \lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) - Q_2(x, \lambda)}{\Pi_{j=1}^n(\lambda + \gamma_j)} \to 0 \quad \text{if} \quad \Re(\lambda) \to -\infty$$

In other words,

(9.2) $$P_2(x, \lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) - Q_2(x, \lambda) = o(\lambda^n)$$

If $m = n$ then $P_2(x, \lambda) \sim \lambda^n$ and we have clearly that $e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) \to 0$ These two facts put together with property (9.2) give us that $Q_2(x, \lambda) = o(\lambda^n)$ and so,

$$\deg Q_2(x, \lambda) \leq n - 1 = m - 1$$

If $m = n + 1$, then $P_2(x, \lambda) \sim \lambda^{n+1}$ and $P_2(x, \lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) = o(\lambda^{n+1})$ then, with property (9.2), we deduce that $Q_2(x, \lambda) = o(\lambda^{n+1})$, in other words,

$$\deg Q_2(x, \lambda) \leq n = m - 1$$

The uniqueness of the 4 polynomials come from the uniqueness of the 6 functions. This finish the proof of proposition 9.3.

Now, when $m = n + 1$ and when looking more precisely the preceding identities, since $\deg Q_2(x, \lambda) \leq n$, we have that $\frac{Q_2(x, \lambda)}{\Pi_{j=1}^n(\lambda+\gamma_j)}$ has a finite limit. Since

$$\frac{P_2(x, \lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) - Q_2(x, \lambda)}{\Pi_{j=1}^n(\lambda + \gamma_j)} \to 0 \qquad \lambda \to -\infty$$

we deduce that

$$\frac{P_2(x, \lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda)}{\Pi_{j=1}^n(\lambda + \gamma_j)} \text{ has a finite limit when } \lambda \to -\infty$$

Since $P_2(x, \lambda) \sim \lambda^{n+1}$ and $\Pi_{j=1}^n(\lambda + \gamma_j) \sim \lambda^n$, we obtain that

$\lambda e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda)$ has a finite limit $(-\check{u}(-x))$ when $\lambda \to -\infty$. That proves proposition 3.1. Now since $\frac{P_2(x,\lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda)-Q_2(x,\lambda)}{\Pi_{j=1}^n(\lambda+\gamma_j)}$ goes to 0(for we deduce that $\frac{Q_2(x,\lambda)}{\lambda^n} \to -\check{u}(-x)$. We state this property in the next lemma. $\qquad \square$

Lemma 9.4.
$$\lim_{|\lambda|\to+\infty} \frac{Q_2(x,\lambda)}{\lambda^n} = -\check{u}(-x)$$

10. COMPUTATION OF A AND $\check{A}$ AND PROOF OF THEOREM 4.1 AND COROLLARY 4.2

Proposition 10.1.
$$A(x,\lambda) = \frac{\begin{vmatrix} 1 & \mathbf{l} \\ e^{-\lambda x}\mathbf{m}(x,\lambda) & \mathbf{W}(x) \end{vmatrix}}{|W(x)|} \quad \text{when } m = n,$$

$$A(x,\lambda) = \frac{\begin{vmatrix} e^{-\lambda x}\mathbf{m}(x,\lambda) & \mathbf{W}(x) \end{vmatrix}}{|\tilde{\mathbf{W}}(x)|} \quad \text{when } m = n+1$$

Proof. Recall the equation of proposition 9.1
$$A(x,\lambda) = \frac{1}{\Pi_1^m(\lambda+\beta_i)}\left(P_1(x,\lambda) + Q_1(x,\lambda)e^{-\lambda x}\check{U}_{[-x,0]}(\lambda)\right)$$

Since $A(x,\lambda)$ is an entire function and because $\deg P_1 = n$ and $\deg Q_1 \leq m-1$ (see proposition 9.3), then $A(x,\lambda)$ is necessarily of the next form

$$A(x,\lambda) = a_0(x) - \sum_1^m a_i(x)\frac{\partial^{m_i}}{(-\partial\beta)^{m_i}}\left[\frac{e^{\beta x}\check{U}_{[-x,0]}(-\beta) - e^{-\lambda x}\check{U}_{[-x,0]}(\lambda)}{\beta+\lambda}\right]_{\beta=\beta_i}$$

$$= a_0(x) - \sum_1^m a_i(x)e^{-\lambda x}\mathbf{m}_i(x,\lambda)$$

for some coefficients $a_i(x)$, $i = 0,1,\ldots,m$
Thus,

$$P_1(x,\lambda) = \Pi_1^m(\lambda+\beta_i)\left(a_0(x) - \sum_1^m a_i(x)\frac{\partial^{m_i}}{(-\partial\beta)^{m_i}}\left[\frac{e^{\beta x}\check{U}_{[-x,0]}(-\beta)}{\lambda+\beta}\right]_{\beta=\beta_i}\right)$$

$$Q_1(x,\lambda) = \Pi_1^m(\lambda+\beta_i)\left(\sum_1^m a_i(x)\frac{\partial^{m_i}}{(-\partial\beta)^{m_i}}\left[\frac{1}{\lambda+\beta}\right]_{\beta=\beta_i}\right)$$

Now using property $P_1(x,\lambda) \sim \lambda^n$ of lemma 9.3, one obtain

(10.1) $$a_0(x) = 1 \quad \text{if} \quad m = n$$

(10.2) $$a_0(x) = 0 \quad \text{and} \quad \sum_{i=1}^m a_i(x)\mathbf{w}_i(x) = -1 \quad \text{if} \quad m = n+1$$

On the other hand, take the identity

$$\check{B}(x,\lambda) = \frac{\check{\psi}(\lambda)}{\Pi(\lambda+\gamma_j)} \cdot \left(P_1(x,\lambda) - Q_1(x,\lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) \right),$$

and replace P_1 and Q_1 by the above expressions, we get

$$\check{B}(x,\lambda) = \check{\psi}(\lambda)\frac{\Pi_1^m(\lambda+\beta_i)}{\Pi_1^n(\lambda+\gamma_j)} \left(a_0(x) - \sum_{i=1}^m a_i(x)\frac{\partial^{m_i}}{(-\partial\beta)^{m_i}} \left[\frac{e^{\beta x}\check{U}_{[-x,0]}(-\beta) + e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda)}{\beta+\lambda} \right) \right]_{\beta=\beta_i} \right)$$

The function $\check{B}(x,\lambda)$ is holomorphic on the left half plan, in particular at points $\lambda = -\gamma_j$, this implies that for every $j = 1,\ldots,n$,

$$\frac{\partial^{n_j}}{(\partial\lambda)^{n_j}} \sum_{i=1}^m a_i(x)\frac{\partial^{m_i}}{(-\partial\beta)^{m_i}} \left[\frac{e^{\beta x}\check{U}_{[-x,0]}(-\beta) + e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda)}{\beta+\lambda} \right) \right]_{\beta=\beta_i,\lambda=-\gamma_j} = \frac{\partial^{n_j}}{(\partial\lambda)^{n_j}}[a_0(x)]_{\lambda=-\gamma_j}$$

In other words,

$$\sum_{i=1}^m a_i(x)W_{i,j}(x) = 1_{n_j=0} \qquad \text{if} \quad m = n$$

And

$$\sum_{i=1}^m a_i(x)W_{i,j}(x) = 0 \qquad \text{if } m = n+1$$

These n equations added to equation (10.2) if $m = n+1$, form a linear system of m equations with variables $a_i(x)$, $i = 1,\ldots,m$. It is easy to be convinced that each solution of this systems brings a new couple of polynomials $P_1(x,\lambda)$ and $Q_1(x,\lambda)$. The uniquiness of such a couple leads to the uniquiness of this solution. Thus the system is a Cramer System and the determinant of the matrix $\mathbf{W}(x)$ for $m = n$ (resp. of the matrix $\tilde{W}(x)$ for $m = n+1$) does not vanish. Finally we obtain,
For $m = n$,

$$a_0(x) = 1 \qquad a_i(x) = \frac{|\mathbf{W}_i^1(x)|}{|\mathbf{W}(x)|} \quad \text{for} \quad i = 1,\ldots,n$$

and

$$A(x,\lambda) = a_0(x) - \sum_1^m a_i(x)e^{-\lambda x}\mathbf{m}_i(x,\lambda) = \frac{\begin{vmatrix} 1 & \mathbf{1} \\ e^{-\lambda x}\mathbf{m}(x,\lambda) & \mathbf{W}(x) \end{vmatrix}}{|\mathbf{W}(x)|}.$$

For $m = n+1$,

$$a_0(x) = 0 \qquad a_i(x) = -\frac{|\mathbf{W}_i(x)|}{|\tilde{\mathbf{W}}(x)|} \quad \text{for} \quad i = 1,\ldots,m,$$

and

$$A(x,\lambda) = -\sum_1^m a_i(x)e^{-\lambda x}\mathbf{m}_i(x,\lambda) = \frac{\begin{vmatrix} e^{-\lambda x}\mathbf{m}(x,\lambda) & \mathbf{W}(x) \end{vmatrix}}{|\tilde{\mathbf{W}}(x)|}$$

$\square$

Proposition 10.2.

$$\check{A}(x,\lambda) = \frac{\begin{vmatrix} \check{U}_{[-x,0]}(\lambda_1) & \mathbf{v}(x) \\ \mathbf{m}(x,\lambda_1) & \mathbf{W}(x) \end{vmatrix}}{|\mathbf{W}(x)|} \quad \textit{if} \quad m = n$$

$$\check{A}(x,\lambda) = \frac{\begin{vmatrix} \check{U}_{[-x,0]}(\lambda_1) & \check{u}(-x) & \mathbf{v}(x) \\ \mathbf{m}(x,\lambda_1) & \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{|\tilde{\mathbf{W}}(x)|} \quad if \quad m = n+1$$

Proof. Take the equation of proposition 9.1,

$$\check{A}(x,\lambda) = \frac{1}{\Pi_1^m(\lambda+\beta_i)}\Big(Q_2(x,\lambda)e^{\lambda x} + P_2(x,\lambda)\check{U}_{[-x,0]}(\lambda)\Big).$$

Since $\check{A}(x,\lambda)$ is an entire function of λ and $P_2(\lambda) \sim \lambda^m$, $\deg Q_2 \le m-1$ according to proposition 9.3, then $\check{A}(x,\lambda)$ is necessarily of the next form

$$\check{A}(x,\lambda) = \check{U}_{[-x,0]}(\lambda) + \sum_1^m \check{a}_i(x)\frac{\partial^{m_i}}{(-\partial\beta)^{m_i}}\Big[\frac{\check{U}_{[-x,0]}(\lambda) - e^{(\lambda+\beta)x}\check{U}_{[-x,0]}(-\beta)}{\lambda+\beta}\Big]_{\beta=\beta_i}$$

$$= \check{U}_{[-x,0]}(\lambda) - \sum_1^m \check{a}_i(x)\mathbf{m}_i(x,\lambda)$$

For some coefficients $\check{a}_i(x)$, $i = 1,\ldots,m$. Thus,

$$Q_2(x,\lambda) = -\Pi_1^m(\lambda+\beta_i)\Big(\sum_1^m \check{a}_i(x)\frac{\partial^{m_i}}{(-\partial\beta)^{m_i}}\Big[\frac{e^{\beta x}\check{U}_{[-x,0]}(-\beta_i)}{\lambda+\beta}\Big]_{\beta=\beta_i}\Big)$$

$$P_2(x,\lambda) = \Pi_1^m(\lambda+\beta_i)\Big(1 + \sum_1^m \check{a}_i(x)\frac{\partial^{m_i}}{(-\partial\beta)^{m_i}}\Big[\frac{1}{\lambda+\beta}\Big]_{\beta=\beta_i}\Big)$$

Take now the identity of proposition 9.1,

$$\check{C}(x,\lambda) = \frac{e^{\lambda x}\check{\psi}(\lambda)}{\Pi(\lambda+\gamma_j)}\Big(P_2(x,\lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) - Q_2(x,\lambda)\Big)$$

$\check{C}(x,\lambda)$ is holomorphic on $\{Re(\lambda < 0\}$. Then for all $j = 1,\ldots,n$,

$$\frac{\partial^{n_j}}{(\partial\lambda)^{n_j}}\Big[\frac{P_2(x,\lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) - Q_2(x,\lambda)}{\Pi_1^m(\lambda+\beta_i)}\Big]_{\lambda=-\gamma_j} = 0$$

In this last equation, when replacing $P_2(x,\lambda)$ and $Q_2(x,\lambda)$ by their above expressions, we obtain the next system, for $j = 1,\ldots,n$,

$$\frac{\partial^{n_j}}{(\partial\lambda)^{n_j}}\Big[e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda)\Big]_{\lambda=-\gamma_j}$$

$$+ \sum_{i=1}^m \check{a}_i(x)\frac{\partial^{n_j}}{(\partial\lambda)^{n_j}}\frac{\partial^{m_i}}{(-\partial\beta)^{m_i}}\Big[\frac{e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) + e^{\beta x}\check{U}_{[-x,0]}(-\beta)}{\lambda+\beta}\Big]_{\beta=\beta_i,\lambda=-\gamma_j} = 0$$

In other words ,

(10.3) $$\sum_{i=1}^m \check{a}_i(x)\mathbf{W}_{i,j}(x) = \mathbf{v}_j(x)$$

This a system of n equations where the variables are the m terms $\check{a}_i(x)$.

When $m = n + 1$, we have the property of lemma 9.4 $\frac{Q_2(x,\lambda)}{\lambda^n} \to -\breve{u}(-x)$ when $\lambda \to -\infty$ then

$$\sum_{i=1}^{m} \breve{a}_i(x) \frac{\partial^{m_i}}{(-\partial\beta)^{m_i}} \left[e^{\beta x} \breve{U}_{[-x,0]}(-\beta) \right]_{\beta=\beta_i} = \breve{u}(-x)$$

In other words

(10.4) $$\sum_{i=1}^{m} \breve{a}_i(x) \mathbf{w}_i(x) = \breve{u}(-x)$$

We have already seen in the previous proof, that $\det \mathbf{W}(x) \neq 0$ if $m = n$ and $\det \tilde{\mathbf{W}}(x) \neq 0$ if $m = n + 1$. Then the system (10.3) augmented by the equation (10.4) if $m = n + 1$, is a Cramer system; and the solution is given as follows. If $m = n$,

$$\breve{a}_i(x) = \frac{|\mathbf{W}_i^{[\mathbf{v}(x)]}(x)|}{|\mathbf{W}(x)|}$$

and

$$\breve{A}(x,\lambda) = \breve{U}_{[-x,0]}(\lambda) - \sum_{1}^{m} \breve{a}_i(x) \mathbf{m}_i(x,\lambda) = \frac{\begin{vmatrix} \breve{U}_{[-x,0]}(\lambda) & \mathbf{v}(x) \\ \mathbf{m}(x,\lambda) & \mathbf{W}(x) \end{vmatrix}}{|\mathbf{W}(x)|}$$

If $m = n + 1$,

$$\breve{a}_i(x) = \frac{|\tilde{\mathbf{W}}(x)|_i^{[\breve{u}(-x),\mathbf{v}(x)]}|}{|\tilde{\mathbf{W}}(x)|}$$

$$\breve{A}(x,\lambda) = \breve{U}_{[-x,0]}(\lambda) - \sum_{1}^{m} \breve{a}_i(x) e^{\lambda x} \mathbf{m}_i(x,\lambda) = \frac{\begin{vmatrix} \breve{U}_{[-x,0]}(\lambda) & \breve{u}(-x) & \mathbf{v}(x) \\ e^{\lambda x}\mathbf{m}(x,\lambda) & \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{|\tilde{\mathbf{W}}(x)|}$$

$\square$

Proof of theorem 4.1 and corollary 4.2
Theorem 4.1 follows from ropositions 9.1, 10.1 and 10.2.
After that, corollary 4.2 becomes obvious by the Laplace inversion of the functions $\mathbf{m}_i(x,\lambda)$ given in proposition 2.6.

11. COMPUTATION OF $\breve{C}(x,\lambda)$ AND $B(x,\lambda)$ AND PROOF OF THEOREMS 5.1 AND COROLLARY 5.2

Proposition 11.1.

$$\breve{C}(x,\lambda) = \breve{\psi}(\lambda) \frac{\begin{vmatrix} \breve{U}_{]-\infty,-x[}(\lambda) & -e^{\lambda x}\mathbf{n}(x,\lambda) \\ \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{|\mathbf{W}(x)|}$$

$$B(x,\lambda) = \frac{\begin{vmatrix} 1 & -\mathbf{i}(\lambda) \\ \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{|\mathbf{W}(x)|} \quad if \quad m = n$$

$$\breve{C}(x,\lambda) = \breve{\psi}(\lambda) \frac{\begin{vmatrix} \lambda\breve{U}_{]-\infty,-x[}(\lambda) & -\breve{U}_{]-\infty,-x[}(\lambda) & e^{\lambda x}\mathbf{n}(x,\lambda) \\ \mathbf{w}'(x) & \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{|\tilde{\mathbf{W}}(x)|}$$

$$B(x,\lambda) = \frac{\begin{vmatrix} \lambda & -1 & \mathbf{i}(\lambda) \\ \mathbf{w}'(x) & \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{|\tilde{\mathbf{W}}(x)|} \quad if \quad m = n+1$$

Proof. We take the expression of $\check{C}(x,\lambda)$ of proposition 9.3,

$$\check{C}(x,\lambda) = \frac{\check{\psi}(\lambda)}{\Pi_1^n(\lambda+\gamma_j)}\Big(-Q_2(x,\lambda)e^{\lambda x} + P_2(x,\lambda)\check{U}_{]-\infty,-x[}(\lambda)\Big)$$

with $\deg P_2 = m$ and $\deg Q_2 \le m-1$.

Because $\check{C}(x,\lambda)$ is holomorphic on $\{\Re(\lambda) < 0\}$, the function $e^{-\lambda x}\check{C}(x,\lambda)$ is in the next form,

$$e^{-\lambda x}\check{C}(x,\lambda) = \check{\psi}(\lambda).\Big(\check{c}_{-1}(x)\lambda e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) + \check{c}_0(x)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) + \check{d}_0(x)$$

$$-\sum_{j=1}^{n}\check{c}_j(x)\frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}}\Big[\frac{e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) - e^{\gamma x}\check{U}_{]-\infty,-x[}(-\gamma)}{\lambda+\gamma}\Big]_{\gamma=\gamma_j}$$

In other words,

$$e^{-\lambda x}\check{C}(x,\lambda)$$

$$= \check{\psi}(\lambda)\Big(\check{c}_{-1}(x)\lambda.e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) + \check{c}_0(x)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) + \check{d}_0(x) - \sum_{j=1}^{n}\check{c}_j(x)\mathbf{n}_j(x,\lambda)\Big)$$

for some coefficients $\check{c}_j(x)$, $j = -1,0,1,\ldots,n$ and a coefficient $\check{d}_0(x)$.

Thus

$$P_2(x,\lambda) = \Pi(\lambda+\gamma_j)\Big(\check{c}_{-1}(x)\lambda + \check{c}_0(x) - \sum_{j=1}^{n}\check{c}_j(x)\frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}}\Big[\frac{1}{\lambda+\gamma}\Big]_{\gamma=\gamma_j}\Big)$$

And

$$Q_2(x,\lambda) = \Pi(\lambda+\gamma_j)\Big(-\check{d}_0(x) - \sum_{j=1}^{n}\check{c}_j(x)\frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}}\Big[\frac{e^{\gamma x}\check{U}_{]-\infty,-x[}(-\gamma)}{\lambda+\gamma}\Big)\Big]_{\gamma=\gamma_j}\Big)$$

Take the equation of proposition 9.1

$$e^{-\lambda x}\check{A}(x,\lambda) = \frac{1}{\Pi(\lambda+\beta_i)}\Big(Q_2(x,\lambda) + P_2(x,\lambda)e^{-\lambda x}\check{U}_{[-x,0]}(\lambda)\Big).$$

The function $\check{A}(x,\lambda)$ is an entire function, then we obtain the equation for every $i = 1,\ldots,m$,

$$\frac{\partial^{m_i}}{(\partial\lambda)^{m_i}}\Big[\frac{Q_2(x,\lambda) + P_2(x,\lambda)e^{-\lambda x}\check{U}_{[-x,0]}(\lambda)}{\Pi_1^n(\lambda+\gamma_j)}\Big]_{\lambda=-\beta_i} = 0$$

Taking the expressions of P_2 and Q_2 upwards, we obtain the next equations,

$$\frac{\partial^{m_i}}{(\partial\lambda)^{m_i}}\Big[\lambda\check{c}_{-1}(x)e^{-\lambda x}\check{U}_{[-x,0]}(\lambda) + \check{c}_0(x)e^{-\lambda x}\check{U}_{[-x,0]}(\lambda) - \check{d}_0(x)\Big]_{\lambda=-\beta_i}$$

$$-\sum_{j=1}^{n}\check{c}_j(x)\Big[\frac{\partial^{m_i}}{(\partial\lambda)^{m_i}}\frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}}\frac{e^{-\lambda x}\check{U}_{[-x,0]}(\lambda) + e^{\gamma x}\check{U}_{]-\infty,-x[}(\gamma)}{\lambda+\gamma}\Big]_{\lambda=-\beta_i,\gamma=\gamma_j} = 0$$

In other words,
(11.1)
$$\check{c}_0(x)\mathbf{w}_i(x) + \sum_{j=1}^{n}\check{c}_j(x)W_{i,j}(x) = \frac{\partial^{m_i}}{(\partial\lambda)^{m_i}}\Big[-\check{c}_{-1}(x)\lambda e^{-\lambda x}\check{U}_{[-x,0]}(\lambda) + \check{d}_0(x)\Big]_{\lambda=-\beta_i}$$

We can compute the coefficients $\check{c}_{-1}(x)$, $\check{c}_0(x)$ and $\check{d}_0(x)$ by using the assertions of lemma 9.3. More precisely, if $m = n$, we have $P_2(x,\lambda) \sim \lambda^n$ and $\deg Q_2 \leq n - 1$, thus we obtain

$$(11.2) \qquad\qquad\qquad \check{c}_{-1}(x) = \check{d}_0(x) = 0 \qquad \check{c}_0(x) = 1$$

If $m = n + 1$ then $P_2(x,\lambda) \sim \lambda^m$ and according to proposition 9.4, $Q_2(x,\lambda) \sim [-\check{u}(-x)]\lambda^n$, thus we obtain

$$(11.3) \qquad\qquad\qquad d_0(x) = \check{u}(-x) \qquad \check{c}_{-1}(x) = 1$$

Then, if $m = n$, using the system (11.1) and equation (11.2) we get the new system

$$\sum_{j=1}^{n}\check{c}_j(x)\mathbf{W}_{i,j}(x) = -\mathbf{w}_i(x) \qquad \text{for } i = 1,\ldots,n$$

Thus

$$\check{c}_j(x) = -\frac{|\mathbf{W}^j_{[\mathbf{w}(x)]}(x)|}{|\mathbf{W}(x)|}$$

And

$$\check{C}(x,\lambda) = \check{\psi}(\lambda).\Big(\check{U}_{]-\infty,-x[}(\lambda) - e^{\lambda x}.\sum_{j=1}^{n}\check{c}_j(x)\mathbf{n}_j(x,\lambda)\Big)$$

$$= \check{\psi}(\lambda)\frac{\begin{vmatrix} \check{U}_{]-\infty,-x[}(\lambda) & -e^{\lambda x}\mathbf{n}(x,\lambda) \\ \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{|\mathbf{W}(x)|}$$

And

$$B(x,\lambda) = \frac{P_2(x,\lambda)}{\Pi(\lambda+\gamma_j)} = 1 - \sum_1^n\check{c}_j(x)\frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}}\Big[\frac{1}{\lambda+\gamma}\Big]_{\gamma=\gamma_j} = 1 - \sum_1^n\check{c}_j(x)\mathbf{i}_j(\lambda)$$

$$= \frac{\begin{vmatrix} 1 & -\mathbf{i}(\lambda) \\ \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{|\mathbf{W}(x)|}$$

Similarely if $m = n + 1$, we obtain from system (11.1) and equation (11.3), the new Cramer system

$$\check{c}_0(x)\mathbf{w}_i(x) + \sum_{j=1}^{n}c_j(x)\mathbf{W}_{i,j}(x) = \frac{\partial^{m_i}}{(-\partial\beta)^{m_i}}\Big[\beta e^{\beta x}\check{U}_{[-x,0]}(-\beta) + \check{u}(-x)\Big]_{\beta=\beta_i} = \mathbf{w}'_i(x)$$

Then (for $j = 0, 1, \ldots, n$),

$$\check{c}_j(x) = \frac{\left| \tilde{\mathbf{W}}(x)^{j+1}_{[\mathbf{w}'(x)]} \right|}{\left| \tilde{\mathbf{W}}(x) \right|}$$

And

$$\check{C}(x,\lambda) = \check{\psi}(\lambda).\Big(\lambda.\check{U}_{]-\infty,-x[}(\lambda) + \check{u}(-x)e^{\lambda x} + \check{c}_0(x)\check{U}_{]-\infty,-x[}(\lambda) - \sum_{j=1}^{n} \check{c}_j(x)e^{\lambda x}\mathbf{n}_j(x,\lambda)\Big)$$

$$= \frac{\left| \begin{matrix} \lambda\check{U}_{]-\infty,-x[}(\lambda) + \check{u}(-x)e^{\lambda x} & -\check{U}_{]-\infty,-x[}(\lambda) & e^{\lambda x}\mathbf{n}(x,\lambda) \\ \mathbf{w}'(x) & \mathbf{w}(x) & \mathbf{W}(x) \end{matrix} \right|}{\left| \tilde{\mathbf{W}}(x) \right|}$$

$$B(x,\lambda) = \frac{P_2(x,\lambda)}{\Pi(\lambda + \gamma_j)} = \lambda + \check{c}_0(x) - \sum_{1}^{n} \check{c}_j(x)\frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}}\Big[\frac{1}{\lambda + \gamma}\Big]_{\gamma=\gamma_j}$$

$$= \lambda + \check{c}_0(x) - \sum_{j=1}^{n} \check{c}_j(x)\mathbf{i}_j(\lambda)$$

$$= \frac{\left| \begin{matrix} \lambda & -1 & \mathbf{i}(\lambda) \\ \mathbf{w}'(x) & \mathbf{w}(x) & \mathbf{W}(x) \end{matrix} \right|}{\left| \tilde{\mathbf{W}}(x) \right|}$$

$\square$

Proof of theorem 5.1 and corollary 5.2.
The proof of theorem 5.1 follows from the above proposition and proposition 9.1. Corollary 5.2 follow then with the help the inverse Laplace transform of the functions $n_j(x,\lambda)$ given in proposition 2.6.

12. Computation of the functions $C(x,\lambda)$ and $\check{B}(x,\lambda)$ and proof of theorem 6.1 and corollary 6.2

Proposition 12.1.

$$C(x,\lambda) = e^{-\lambda x}\frac{\left| \begin{matrix} 0 & \mathbf{i}(\lambda) \\ \mathbf{c} & \mathbf{W}(x) \end{matrix} \right|}{|\mathbf{W}(x)|} \qquad \check{B}(x,\lambda) = \check{\psi}(\lambda)\frac{\left| \begin{matrix} 1 & \mathbf{n}(x,\lambda) \\ \mathbf{c} & \mathbf{W}(x) \end{matrix} \right|}{|\mathbf{W}(x)|} \quad if \quad m = n$$

$$C(x,\lambda) = e^{-\lambda x}\frac{\left| \begin{matrix} 0 & 1 & \mathbf{i}(\lambda) \\ \mathbf{c} & \mathbf{w}(x) & \mathbf{W}(x) \end{matrix} \right|}{|\tilde{\mathbf{W}}(x)|}$$

$$\check{B}(x,\lambda) = \check{\psi}(\lambda)\frac{\left| \begin{matrix} 0 & e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) & \mathbf{n}(x,\lambda) \\ \mathbf{c} & \mathbf{w}(x) & \mathbf{W}(x) \end{matrix} \right|}{|\tilde{\mathbf{W}}(x)|} \quad if \quad m = n+1$$

Proof. According to proposition 9.3, we have the identity

$$\check{B}(x,\lambda) = \frac{\check{\psi}(\lambda)}{\Pi_1^n(\lambda + \gamma_j)}.\Big(P_1(x,\lambda) - Q_1(x,\lambda)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda)\Big)$$

with

$$\deg P_1(\lambda) \sim \lambda^n \quad and \quad \deg Q_1 \leq m - 1$$

Since $\check{B}(x,\lambda)$ is holomorphic on $\{\Re(\lambda) < 0\}$, we can write $\check{B}(x,\lambda)$ in the following form,

$$\check{B}(x,\lambda) = \check{\psi}(\lambda)\left(1+\check{b}_0(x)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda)+\sum_{j=1}^{n}\check{b}_j(x)\frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}}\left[\frac{e^{\gamma x}\check{U}_{]-\infty,-x[}(-\gamma)-e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda)}{\lambda+\gamma}\right]_{\gamma=\gamma_j}\right)$$

In other words,

$$\check{B}(x,\lambda) = \check{\psi}(\lambda).\left(1 + \check{b}_0(x)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) - \sum_{j=1}^{n}\check{b}_j(x)\mathbf{n}_j(x,\lambda)\right)$$

for some coefficients $\check{b}_j(x)$, $j = -1, 0, 1, \ldots, n$ and a coefficient $\check{d}_0(x)$.
Thus the polynomial P_1 and Q_1 are given by the next expressions

$$P_1(x,\lambda) = \Pi_1^n(\lambda + \gamma_j)\left(1 + \sum_{j=1}^{n}\check{b}_j(x)\frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}}\left[\frac{e^{\gamma x}\check{U}_{]-\infty,-x[}(-\gamma)}{\lambda+\gamma}\right]_{\gamma=\gamma_j}\right)$$

$$Q_1(x,\lambda) = \Pi_1^n(\lambda + \gamma_j)\left(-\check{b}_0(x) + \sum_{j=1}^{n}\check{b}_j(x)\frac{\partial^{n_j}}{(-\partial\gamma)^{n_j}}\left[\frac{1}{\lambda+\gamma}\right]_{\gamma=\gamma_j}\right)$$

On the other hand, according to proposition 9.1, we have

$$A(x,\lambda) = \frac{1}{\Pi_1^m(\lambda + \beta_i)}\left(P_1(x,\lambda) + Q_1(x,\lambda)e^{-\lambda x}\check{U}_{[-x,0]}(\lambda)\right)$$

Since $A(x,\lambda)$ is an entire function of λ, then it is holomorphic at $\lambda = -\beta_i$; Thus we have the equations for $i = 1, \ldots, m$.

$$\frac{\partial^{m_i}}{(\partial\lambda)^{m_i}}\left[\frac{P_1(x,\lambda) + Q_1(x,\lambda)e^{-\lambda x}\check{U}_{[-x,0]}(\lambda)}{\Pi_1^n(\lambda + \gamma_j)}\right]_{\lambda=-\beta_i} = 0,$$

that is,

$$\check{b}_0(x)\frac{\partial^{m_i}}{(\partial\lambda)^{m_i}}\left[e^{-\lambda x}\check{U}_{[-x,0]}(\lambda)\right]_{\lambda=-\beta_i}$$

$$-\sum_{j=1}^{n}\check{b}_j(x)\frac{\partial^{m_i}\partial^{n_j}}{(\partial\lambda)^{m_i}(-\partial\gamma)^{n_j}}\left[\frac{e^{\gamma x}\check{U}_{]-\infty,-x[}(-\gamma)+e^{-\lambda x}\check{U}_{[-x,0]}(\lambda)}{\lambda+\gamma}\right]_{\gamma=\gamma_j,\lambda=-\beta_i} = 1_{m_i=0}$$

In other words,

$$(12.1) \qquad \check{b}_0(x)\mathbf{w}_i(x) + \sum_{j=1}^{n}\check{b}_j(x)\mathbf{W}_{i,j}(x) = 1_{m_i=0}$$

Moreover, if $m = n$, since $\deg Q_1(x,\lambda) \leq n-1$, we have

$$(12.2) \qquad \check{b}_0(x) = 0$$

We have already seen that the determinants $|\mathbf{W}(x)|$ (if $m = n$) and $|\tilde{\mathbf{W}}(x)|$ (if $m = n+1$), then the system (12.1) augmented with (12.2) if $m = n$, form a Cramer system whose solutions are given by the next expressions,
If $m = n$,

$$\check{b}_0(x) = 0 \qquad \check{b}_j(x) = \frac{|\mathbf{W}_j^c(x)|}{|\mathbf{W}(x)|} \quad \text{for } j = 1, \ldots, n$$

Thus

$$\check{B}(x,\lambda) = \check{\psi}(\lambda)(1 - \sum_1^n \check{b}_j(x)\mathbf{n}_j(x,\lambda)) = \check{\psi}(\lambda)\frac{\begin{vmatrix} 1 & \mathbf{n}(x,\lambda) \\ \mathbf{c} & \mathbf{W}(x) \end{vmatrix}}{|\mathbf{W}(x)|}$$

and the expression of $C(x,\lambda)$ given in proposition 9.3 gives

$$C(x,\lambda) = -\frac{Q_1(x,\lambda)e^{-\lambda x}}{\Pi_1^n(\lambda+\gamma_j)} = e^{-\lambda x}\left(\sum_1^n \frac{-\check{b}_j(x)}{\lambda+\gamma_j}\right) = e^{-\lambda x}\frac{\begin{vmatrix} 0 & \mathbf{i}(\lambda) \\ \mathbf{c} & \mathbf{W}(x) \end{vmatrix}}{|\mathbf{W}(x)|}$$

If $m = n+1$, we have for $j = 0, \ldots, n$,

$$\check{b}_j(x) = \frac{\left|\tilde{\mathbf{W}}^{\mathbf{c}}_{j+1}(x)\right|}{\left|\tilde{\mathbf{W}}(x)\right|}$$

And

$$\check{B}(x,\lambda) = \check{\psi}(\lambda)\left(1 - \check{b}_0(x)e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) - \sum_1^n \check{b}_j(x)\mathbf{n}_j(x,\lambda)\right)$$

$$= \check{\psi}(\lambda)\frac{\begin{vmatrix} 1 & e^{-\lambda x}\check{U}_{]-\infty,-x[}(\lambda) & \mathbf{n}(x,\lambda) \\ \mathbf{c} & \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{\left|\tilde{\mathbf{W}}(x)\right|}$$

and then,

$$C(x,\lambda) = -\frac{Q_1(x,\lambda)e^{-\lambda x}}{\Pi_1^n(\lambda+\gamma_j)} = e^{-\lambda x}\left(\check{b}_0(x) - \sum_1^n \frac{\check{b}_j(x)}{\lambda+\gamma_j}\right)$$

$$= e^{-\lambda x}\frac{\begin{vmatrix} 0 & 1 & \mathbf{i}(\lambda) \\ \mathbf{c} & \mathbf{w}(x) & \mathbf{W}(x) \end{vmatrix}}{\left|\tilde{\mathbf{W}}(x)\right|}$$

$$\square$$

Proof of corollary 6.2. This corollary becomes obvious when using propositions 9.1 and the previous one and when using the Laplace thransform inversion of $\mathbf{n}_j(x,\lambda)$ and $\mathbf{i}_j(\lambda)$ given in proposition 2.6

13. Proof of theorems 7.1 and 8.1

It has been proven in [F10] (propositions 6.3. and 6.4) that the distributions of the triple (X,I,S) at time U_x and at time T_a^b are also characterized by the 6 functions $A, \check{A}, B, \check{B}, C, \check{C}$. Then, we translate these two propositions with reference to the explicit forms of the functions obtained previously.

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